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QUANTIZED HYPERALGEBRAS OF RANK 1

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ABSTRACT

We study the algebra U_ζ obtained via Lusztig's 'integral' form [Lu 1, 2] of the generic quantum algebra for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ modulo the two-sided ideal generated by $K^l - 1$. We show that U_ζ is a smash product of the quantum deformation of the restricted universal enveloping algebra $\mathbf{u}_\zeta$ of $\mathfrak{g}$ and the ordinary universal enveloping algebra U of $\mathfrak{g}$, and we compute the primitive (= prime) ideals of U_ζ. Next we describe a decomposition of $\mathbf{u}_\zeta$ into the simple U-submodules, which leads to an explicit formula for the center and the indecomposable direct summands of U_ζ. We conclude with a description of the lattice of cofinite ideals of U_ζ in terms of a unique set of lattice generators.

0. Introduction

G. Lusztig constructed in [Lu 1,2,3] quantum algebras associated to the defining relations of the finite-dimensional semi-simple Lie algebra $\mathfrak{g}$. The method used there is similar to the one employed by Kostant [Ko] in his construction of the hyperalgebra for $\mathfrak{g}$. We refer to Lusztig's algebra as the quantum hyperalgebra of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{sl}_2$ be the rank 1 simple Lie algebra. Fix a field $\mathbb{K}$ of characteristic zero containing a primitive ℓ-th root of unity ζ of an odd order. We let U_q stand for the usual generic quantum algebra of $\mathfrak{sl}_2$. We let $\hat{U}_\zeta$ denote the quantum algebra associated with $\mathfrak{sl}_2$ as in [Lu 1]. We define U_ζ as the quotient of $\hat{U}_\zeta$

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modulo the ideal generated by $K^\ell - 1$. We denote by $\mathbf{u}_\zeta$ the Frobenius–Lusztig kernel in U_ζ and let U stand for the ordinary enveloping algebra of $\mathfrak{sl}_2$ over $\mathbb{K}$.

The goal of this paper is to obtain an explicit description of the primitive ideals, the center, blocks and the lattice of cofinite ideals of U_ζ.

The paper is organized as follows. In Section 1 we show that U_ζ is the smash product of the Frobenius–Lusztig kernel and U. This feature of U_ζ is one that distinguishes this case from the higher rank cases, and informs the structure of U_ζ. The theory of simple modules and their annihilators is taken up in Section 2. Here we point out that the prime ideals are primitive. The main result is the existence of the Steinberg–Lusztig factorization [Lu 1] for every (not necessarily finite-dimensional) simple U_ζ-module and an explicit formula for a primitive ideal. As a technical preliminary we give the presentation of the "diagonal" part U_ζ^0 of U_ζ by generators and relations. The results of this section admit of a generalization to all semi-simple finite-dimensional $\mathfrak{g}$. This has been carried out in [CK 2]. However, in the present case the proof proceeds along different lines which yield a stronger result.

In Section 3 we describe the decomposition of $\mathbf{u}_\zeta$ into a direct sum of simple U-modules. The exposition relies heavily on the structure of principal indecomposable modules for $\mathbf{u}_\zeta$. In Section 4 we compute the center of U_ζ. As a consequence we show that every indecomposable direct summand (block) of U_ζ is of the form ϵU_ζ where ϵ is a block idempotent of $\mathbf{u}_\zeta$.

Section 5 contains a description of the lattice of cofinite ideals. An important feature of the lattice is its distributivity. Consequently it has a unique set of lattice generators, namely, the meet-irreducible ideals. We show that each such ideal is the annihilator of a simple, Weyl or co-Weyl module, or else the annihilator of the injective hull of a simple finite-dimensional module, viewed as comodule for the finitary dual of U_ζ. For a more precise statement on the structure of these ideals, see Theorem 7.

It should be noted that the complete lattice of two-sided ideals of U_q is due to [Ba]. His classification asserts in part that a cofinite ideal is a unique product of maximal ideals. This can be seen directly as follows. Let I be such an ideal. Then U_q/I is finite dimensional, hence a semisimple U_q-module. Therefore $I = \bigcap \mathfrak{m}_r$, where the $\mathfrak{m}_r$ run over all maximal ideals containing I. But, as $\mathrm{Ext}^1_U(X, Y) = 0$ for all finite dimensional U_q-modules X and Y, a result in [Mo 1] yields $\bigcap \mathfrak{m}_r = \prod \mathfrak{m}_r$.

The main theorems of Section 5 are parallel to the just-mentioned result of [Ba]. A description of the entire lattice of two-sided ideals remains an open question, which will be addressed in a later paper.

We fix some more notation. The comultiplication in U_ζ and U will be denoted by Δ_ζ and Δ, respectively. An element of the form $\left[{K;c \atop t}\right]$ [Lu 1] defined in U_q will be marked by a subscript q. Unsubscribed $\left[{K;c \atop t}\right]$ is $\left[{K;c \atop t}\right]\otimes 1$. An unmodified $\otimes$ means $\otimes_{\mathbb{K}}$. We recall the definition of the quantum Casimir element. This is

$$c_\zeta = FE + \frac{\zeta K + \zeta^{-1}K^{-1}}{(\zeta - \zeta^{-1})^2}.$$

It is straightforward to check that c_ζ lies in the center of U_ζ

1. Smash product representation of U_ζ

In what follows we put $\mathfrak{g}$ equal to $\mathfrak{sl}_2$ and denote by $\{e, h, f\}$ its standard basis.

By [Lu 3] there exists a unique Hopf algebra map $\mathrm{Fr}\colon U_\zeta \to U$, called the Frobenius map, specified on generators of U_ζ as follows:

$$\mathrm{Fr}(E) = \mathrm{Fr}(F) = 0, \mathrm{Fr}(K) = 1, \mathrm{Fr}(E^{(\ell)}) = e \quad \text{and} \quad \mathrm{Fr}(F^{(\ell)}) = f.$$

Let $\mathbf{u}_\zeta$ be the subalgebra of U_ζ generated by E, F and K. Put $\mathbf{u}_\zeta^+ = \mathbf{u}_\zeta \cap \mathrm{Ker}\,\epsilon$. Combining results of [Lu 3] and [An] we know that Ker Fr is generated by $\mathbf{u}_\zeta^+$ as a left or right ideal. Moreover, Fr induces a right U-comodule algebra structure on U_ζ via $\rho\colon U_\zeta \to U_\zeta \otimes U$, $\rho = (I \otimes \mathrm{Fr})\Delta_\zeta$, where I is the appropriate identity map. We conclude using a couple of Schneider's results [Sch 1,2] that $\mathbf{u}_\zeta = U_\zeta^{\mathrm{co}\,U}$. The upshot of these remarks is that we have an exact sequence in the category of Hopf algebras

$$\mathbb{K} \longrightarrow \mathbf{u}_\zeta \longrightarrow U_\zeta \xrightarrow{\mathrm{Fr}} U \longrightarrow \mathbb{K}.$$

We want to show that the above sequence splits by an algebra and right U-comodule homomorphism $\gamma\colon U \to U_\zeta$ satisfying $\mathrm{Fr} \circ \gamma = I$.

Lemma 1: *The mapping γ defined on $\mathfrak{g}$ by $\gamma(e) = E^{(\ell)}, \gamma(f) = F^{(\ell)}$ extends to an algebra homomorphism $U \to U_\zeta$*

Proof: Let $H = [E^{(\ell)}, F^{(\ell)}]$. We must show that $[H, E^{(\ell)}] = 2E^{(\ell)}$ and $[H, F^{(\ell)}] = -2F^{(\ell)}$. To this end we recall that by [Lu 1] we have

$$H = \left[{K \atop \ell}\right] + f, \quad \text{where } f = \sum_{1 \le t \le \ell - 1} F^{(\ell-t)}\left[{K; -2(\ell-t) \atop t}\right]E^{(\ell-t)}.$$

Since

$$F^{(s)}\left[{K;c \atop t}\right] = \left[{K; c+2s \atop t}\right]F^{(s)}$$

it follows that

$$F^{(\ell-t)}\begin{bmatrix} K; -2(\ell-t) \\ t \end{bmatrix} E^{(\ell-t)} = \begin{bmatrix} K \\ t \end{bmatrix} F^{(\ell-t)}E^{(\ell-t)}.$$

An easy induction on s leads to the formula

$$F^{(s)}E^{(s)} = \frac{1}{[s]!^2}\prod_{i=1}^{s}\Big(c_\zeta - \frac{\zeta^{2(i-1)+1}K + \zeta^{-2(i-1)-1}K^{-1}}{(\zeta-\zeta^{-1})^2}\Big). \tag{1.1}$$

Thus f lies in the subalgebra of $\mathbf{u}_\zeta$ generated by K and c_ζ which is centralized by $E^{(\ell)}$ and $F^{(\ell)}$. It follows that

$$\begin{aligned}[H, E^{(\ell)}] &= \Big[\begin{bmatrix} K \\ \ell \end{bmatrix}, E^{(\ell)}\Big] \\ &= E^{(\ell)}\Big\{\begin{bmatrix} K; 2\ell \\ \ell \end{bmatrix} - \begin{bmatrix} K \\ \ell \end{bmatrix}\Big\}.\end{aligned}$$

Applying [Lu 2, (g9)] we obtain in $U_{\mathcal{A}}$

$$\begin{bmatrix} K; 2\ell \\ \ell \end{bmatrix}_q = \sum_{0\le j\le \ell} q^{2\ell(\ell-j)}\begin{bmatrix} 2\ell \\ j \end{bmatrix} K^{-j}\begin{bmatrix} K \\ \ell-j \end{bmatrix}_q$$

which, in view of $[{2\ell \atop j}]_\zeta = 0$ for $0 < j < \ell$, reduces to

$$\begin{bmatrix} K; 2\ell \\ \ell \end{bmatrix} = \begin{bmatrix} K \\ \ell \end{bmatrix} + \begin{bmatrix} 2\ell \\ \ell \end{bmatrix} = \begin{bmatrix} K \\ \ell \end{bmatrix} + 2$$

in U_ζ. Thus we arrive at $[H, E^{(\ell)}] = 2E^{(\ell)}$. Similar computations give $[H, F^{(\ell)}] = -2F^{(\ell)}$. ∎

THEOREM 1: *$U_\zeta = \mathbf{u}_\zeta \# U$ is the smash product of $\mathbf{u}_\zeta$ and U.*

Proof: As γ is an algebra map, it is convolution invertible. In fact, it is also a U-comodule map, i.e., $\rho\circ\gamma = (\gamma\otimes I)\circ\Delta$. Since the mappings in the last equation are algebra homomorphisms, it suffices to check it on generators, which is straightforward. By [DT], γ induces U-action in $\mathbf{u}_\zeta$ via $v\cdot a = \gamma(v_1)a\gamma^{-1}(v_2), v \in U, a \in \mathbf{u}_\zeta$ (where Δv is written as $v_1 \otimes v_2$). This action gives rise to the smash product $\mathbf{u}_\zeta\#U$, which is isomorphic to U_ζ by op. cit. ∎

2. Simple modules and primitive ideals

The subalgebra U_ζ^0 of U_ζ is by definition generated by K and the elements $\left[{K;c \atop t}\right], c \in \mathbb{Z}, t \in \mathbb{Z}_+$ [Lu 1,2]. It is responsible for the weight theory in the category of U_ζ-modules. We are interested in finding generators and relations for U_ζ^0. The following result appears in [CK 2]. We include its proof for the reader's convenience

Proposition 1: *U_ζ^0 is a commutative algebra generated by K and $\left[{K \atop \ell}\right]$ subject to the relation $K^\ell = 1$.*

Proof: We apply the Doi–Takeuchi theory of cleft extensions to U_ζ^0. We remark that U_ζ^0 is a subcoalgebra of U_ζ, and that Fr sends U_ζ^0 to U^0, the subalgebra of U generated by h. It follows that ρ restricted to U_ζ^0 induces a right U^0-comodule structure on the latter. We define an algebra homomorphism ϕ: $U^0 \to U_\zeta^0$ by sending h to $\left[{K \atop \ell}\right]$. We observe that ϕ splits Fr as a U^0-comodule map.

We conclude by [DT] that $U_\zeta^0 = A \#_\sigma U^0$, where $A = (U_\zeta^0)^{\mathrm{co}\, U^0}$. Further, since ϕ is an algebra map, the cocycle σ is trivial. Also U_ζ^0 is commutative, therefore the action of U^0 is trivial as well. Thus $U_\zeta^0 = A \otimes U^0$. It remains to identify A. Let $\mathbf{u}_\zeta^0 = \mathbb{K}[K]$ be the subalgebra of generated by K. We have that A is the subalgebra of coinvariants in U_ζ^0, hence $A = \mathbf{u}_\zeta \cap U_\zeta^0$. Using the PBW-theorem for U_ζ one can see readily that $A = \mathbf{u}_\zeta^0$, and the proof is complete. ∎

Remark 1: We sketch a direct proof of the proposition. First of all we have from [Lu 2] that U_ζ^0 is spanned by the set $\{K^\delta\left[{K \atop t}\right] | \delta = 0, 1; t \geq 0\}$. The relation $K^\ell = 1$ enables us to reduce that set to the subset $\{\left[{K \atop t}\right] | t \geq 0\}$. The last step is a formula of independent interest (cf. [Lu 1, 4.3]).

Lemma 1′: *Let m be a positive integer written as $m = m_0 + \ell m_1, 0 \leq m_0 \leq \ell - 1$. Then $\left[{K \atop m}\right] = \left[{K \atop m_0}\right]\binom{\left[{K \atop \ell}\right]}{m_1}$, where the second factor is the ordinary binomial expression.* ∎

Remark 2: The lemma fails without the assumption $K^\ell = 1$.

The Proposition follows readily. For, the sets

$$\left\{\left[{K \atop a}\right] | 0 \leq a \leq \ell - 1\right\} \quad \text{and} \quad \left\{\binom{\left[{K \atop \ell}\right]}{b} | b \geq 0\right\}$$

form bases for $\mathbf{u}_\zeta^0$ and the subalgebra generated by $\left[{K \atop \ell}\right]$, respectively.

Remark 3: According to the Cartier–Kostant–Milnor–Moore theorem (see [Mo, 5.6.5]) $U_\zeta^0 = \mathbb{K}G \otimes U(P)$, where G is the set of group-like and P is the Lie algebra of primitive elements in U_ζ^0. The Frobenius map restricted to $U(P)$ induces an isomorphism of $U(P)$ with U^0. Hence $\dim P = 1$ and there is a unique primitive d such that $\mathrm{Fr}(d) = h$. We want to give an explicit formula for d. $\mathbf{u}_\zeta^0 = \mathbb{K}[K]$ has ℓ minimal idempotents $e_i, 0 \le i \le \ell - 1$. Explicitly

$$e_m = \frac{1}{\ell}\Big(\sum_{j=0}^{\ell-1} \zeta^{-mj} K^j\Big).$$

It follows by a direct computation that $\Delta_\zeta e_m = \sum_{i+j\equiv m} e_i \otimes e_j$, where the $\equiv$ denotes congruence modulo ℓ. On the other hand, one can verify that

$$\Delta_\zeta\Big(\begin{bmatrix} K \\ \ell \end{bmatrix}\Big) = \begin{bmatrix} K \\ \ell \end{bmatrix} \otimes 1 + 1 \otimes \begin{bmatrix} K \\ \ell \end{bmatrix} + \sum_{i+j\ge\ell} e_i \otimes e_j.$$

We now define d by the formula

$$d = \begin{bmatrix} K \\ \ell \end{bmatrix} + \frac{1}{\ell}\Big(\sum_{m=1}^{\ell-1} m e_m\Big).$$

Using the above formulas it is straightforward to check that d is indeed a primitive element. For an alternate treatment see [CK 2]. ∎

We now turn to a computation of characters of U_ζ^0. Let $X = \mathrm{Alg}(U_\zeta^0, \mathbb{K})$ be the group of algebra homomorphisms under convolution. To every $\chi \in X$ we associate its weight $\lambda = (r, \alpha)$ according to the equalities $\chi(K) = \zeta^r$ and $\chi(\left[\begin{smallmatrix} K \\ \ell \end{smallmatrix}\right]) = \alpha$.

Let $\Lambda \simeq \mathbb{Z}_\ell \times \mathbb{K}$ be the set of all weights. We remark that every integer m can be viewed as a weight. For, writing (uniquely) $m = m_0 + m'\ell$, $0 \le m_0 \le \ell - 1$, we can identify m with the pair (m_0, m'). The next lemma has been proved in [CK 2]. We give a proof for the sake of completeness.

Lemma 2: *Let $\lambda = (r, \alpha)$ and $\mu = (s, \beta)$ be two weights. Then*

$$\lambda + \mu = \begin{cases} (r+s, \alpha+\beta), & \text{if } r+s<\ell, \\ (r+s-\ell, \alpha+\beta+1), & \text{else.} \end{cases}$$

Proof: Pick two characters f and g of weight λ and μ, respectively. Since K is a group-like $(f * g)(K) = \zeta^{r+s}$. Further, recall the formula [CP, 11.2]

$$\Delta\Big(\begin{bmatrix} K \\ \ell \end{bmatrix}\Big) = \sum_{0\le j\le\ell} \begin{bmatrix} K \\ \ell - j \end{bmatrix} K^{-j} \otimes \begin{bmatrix} K \\ j \end{bmatrix} K^{\ell-j}.$$

In view of $K^\ell = 1$ and $f([{K \atop m}]) = [{r \atop m}]_\zeta$ for every $m < \ell$, and similarly for g, it follows readily that

$$(f * g)\left(\begin{bmatrix} K \\ \ell \end{bmatrix}\right) = \alpha + \sum{}' + \beta \quad \text{where } \sum{}' = \sum_{1\le j\le \ell} \zeta^{-(r+s)j} \begin{bmatrix} r \\ \ell - j \end{bmatrix}_\zeta \begin{bmatrix} s \\ j \end{bmatrix}_\zeta .$$

It remains to notice that as $[{m \atop \ell}]_\zeta = 0$ for all $m < \ell$, the identity [Lu, 2(g 9-10)] shows that $\sum' = [{r+s \atop \ell}]$, and the assertion follows from [Lu, 1(3.2)]. ∎

As in the classical case, we define a partial order on Λ by saying $\lambda \le \mu \iff \lambda = \mu - 2n$ for some $n \in \mathbb{Z}^+$. Let $\mathbb{K}_\lambda$ be the 1-dimensional U^0_ζ-module of weight $\lambda = (r, \alpha)$, i.e., $\mathbb{K}_\lambda = \mathbb{K}v_\lambda$ with $Kv_\lambda = \zeta^r v_\lambda$ and $[{K \atop \ell}]v_\lambda = \alpha v_\lambda$. We put $B^+ = U^0_\zeta U^+_\zeta$ and observe that the U^0_ζ action on $\mathbb{K}_\lambda$ can be lifted to the B^+-action by setting $E^{(n)}v_\lambda = 0$ for all $n > 0$. We define (as usual) the U_ζ-module $V(\lambda)$ by $V(\lambda) = U_\zeta \otimes_{B^+} \mathbb{K}_\lambda$ with the left regular action of U_ζ. One can check easily that $F^{(n)}v_\lambda$ has weight $\lambda - 2n$. By the standard argument $V(\lambda)$ has a unique maximal submodule and a unique irreducible quotient denoted by $L(\lambda)$.

In preparation for the next statement we fix some notation. For a U-module M we denote by M^{Fr} the U_ζ-module obtained via the pull-back along Fr. For a weight $\lambda = (m_0, m_1)$ with $m_1 \in \mathbb{Z}$ we write $\lambda = m$, where $m = m_0 + m_1\ell$. If $m_1 = 0$ we say that λ is **ℓ-restricted**. We abbreviate $L(m_0, m_1)$ to $L(m)$. For an $\alpha \in \mathbb{K}$ we let $\overline{L}(\alpha)$ denote the highest weight α simple U-module. For a weight λ we denote by $P(\lambda)$ the primitive ideal

$$P(\lambda) = \mathrm{ann}_{U_\zeta} L(\lambda).$$

If $\lambda = (r, \alpha)$ we set

$$\mathfrak{p}(r) = \mathrm{ann}_{\mathbf{u}_\zeta} L(r) \quad \text{and} \quad \overline{P}(\alpha) = \mathrm{ann}_U \overline{L}(\alpha).$$

THEOREM 2: (i) *Every prime ideal of U_ζ is primitive.*

(ii) *Every primitive ideal of U_ζ has the form $P(\lambda)$ for some weight λ. Furthermore, if $\lambda = (r, \alpha)$ then we have $P(\lambda) = \mathfrak{p}(r)U + \mathbf{u}_\zeta \overline{P}(\alpha)$.*

(iii) *Every simple U_ζ-module S has the Steinberg–Lusztig factorization*

$$S \simeq L(r) \otimes \overline{S}^{\mathrm{Fr}}$$

for a suitable restricted r and a U-module $\overline{S}$. Further, $L(r) \otimes \overline{L}(\alpha)^{\mathrm{Fr}}$ is isomorphic to $L(r, \alpha)$.

First a Lemma.

LEMMA 3: *Let N be the nilpotent radical of $\mathbf{u}_\zeta$. Then N is stable under U and the U-module $\mathbf{u}_\zeta/N$ is trivial.*

Proof: By [Lu 1, 7.1] the restriction of $L(r)$ to $\mathbf{u}_\zeta$ is a simple module for every restricted r. In fact, these exhaust all simple $\mathbf{u}_\zeta$-modules which can be seen directly or following the argument of the classical modular case [Cu]. Thus $N = \bigcap_{0\le r\le \ell}\mathfrak{p}(r)$. Now, U-invariance of every prime ideal of $\mathbf{u}_\zeta$ follows from [Ch 1] or, more immediately, from [GW, Prop. 1.1] in view of the fact that every prime of $\mathbf{u}_\zeta$ is a minimal prime.

Since $\mathbf{u}_\zeta$ is generated by K, E, F it suffices to show that $[E^{(\ell)}, X]$ and $[F^{(\ell)}, X]$ lie in N for every generator X. This is clear for $X = K$ as K commutes with $E^{(\ell)}$ and $F^{(\ell)}$. Suppose $X = F$. Then we have

$$[E^{(\ell)}, F] = \begin{bmatrix} K; -\ell+1 \\ 1 \end{bmatrix} E^{\ell-1}.$$

Also, $E^{\ell-1}L(r) = 0$ for all $r < \ell-1$. Thus $[E^{(\ell)}, F]$ lies in every $\mathfrak{p}(r)$ for $r \neq \ell-1$. Let $r = \ell-1$. Recall that $L(\ell-1)$ is the span of $\{F^i v_0 | 0 \le i \le \ell-1\}$, where v_0 is a primitive vector of weight $\ell-1$. It follows that $E^{\ell-1}L(\ell-1) = \mathbb{K}v_0$. As $\left[\begin{smallmatrix} K;-\ell+1 \\ 1 \end{smallmatrix}\right]v_0 = 0$, the proof is complete. ∎

We proceed to the proof of the theorem. (i) For every U-invariant ideal I of $\mathbf{u}_\zeta$, it is straightforward to check using the smash product structure that $UI = IU$. Hence NU is a nilpotent ideal of U_ζ. In view of the above lemma U_ζ/NU is semisimple, and hence NU is the Jacobson radical of U_ζ.

Now, let P be a prime ideal of U_ζ. By the opening remark $P \supset NU$. Passing on to $U_\zeta/NU \simeq \mathbf{u}_\zeta/N \otimes U \simeq \oplus M_t(U)$ we may assume that P is a prime ideal of $M_t(U)$ for some t. It follows readily that $P = M_t(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of U. Then by [NG] $\mathfrak{p}$ is primitive, and by [Po] so is P.

(ii) Pick P, a primitive ideal of U_ζ, and a simple module S with $P = \operatorname{ann}_{U_\zeta} S$. Let $\mathfrak{p} = P \cap \mathbf{u}_\zeta$. Then $\mathfrak{p} \supset N$, hence $\mathfrak{p}$ is a prime ideal of $\mathbf{u}_\zeta$. For, every ideal of $\mathbf{u}_\zeta$ containing $\mathfrak{p}$ is U-invariant. Therefore, the inclusion $I \cdot J \subset \mathfrak{p}, I, J$ ideals of $\mathbf{u}_\zeta$ implies $IU \cdot JU \subset P$, hence I or J lies in $\mathfrak{p}$.

Choose r such that $\mathfrak{p} = \mathfrak{p}(r)$ and let $\pi\colon U_\zeta \longrightarrow \mathbf{u}_\zeta/\mathfrak{p} \otimes U$ be the map composed of the natural epimorphism $U_\zeta \longrightarrow U_\zeta/\mathfrak{p}U$ and the isomorphism $U_\zeta/\mathfrak{p}U \simeq \mathbf{u}_\zeta/\mathfrak{p} \otimes U$ sending $x\#y + \mathfrak{p}U$ to $(x+\mathfrak{p}) \otimes y$. We remark that $\mathbf{u}_\zeta/\mathfrak{p}$ is just $\operatorname{End}_{\mathbb{K}} L(r)$ because $L(r)$ is absolutely simple. Indeed, the standard argument [Lu 1] for simplicity of $L(r)$ works for every field containing $\mathbb{Q}(\zeta)$. Denoting $\mathbf{u}_\zeta/\mathfrak{p}$ by E we see that S is an $E \otimes U$-module. Now, the standard argument with minor modifications as e.g. in [St, Lemma 68] shows that $S = L(r) \otimes \overline{S}$ for a suitable U-module $\overline{S}$.

Let $\overline{P}$ denote the $\mathrm{ann}_U \overline{S}$. $E \otimes U/\overline{P}$ acts faithfully in S, which implies that $E \otimes \overline{P}$ is the $\mathrm{ann}_{E\otimes U} S$. We want to compute $T := \pi^{-1}(E \otimes \overline{P})$. To this end we notice the equalities $\mathbf{u}_\zeta \cap \mathfrak{p}U = \mathfrak{p}$ and $\overline{P} \cap \mathfrak{p}U = (0)$, both of which come directly from the $\mathbb{K}$-isomorphism $U_\zeta \simeq \mathbf{u}_\zeta \otimes U$. But then $\pi(\mathbf{u}_\zeta \overline{P}) = E \otimes \overline{P}$, which shows that $T = \mathfrak{p}U + \mathbf{u}_\zeta \overline{P}$. On the other hand, $\pi(P) = \mathrm{ann}_{E\otimes U} S$ and therefore $P = T$. Further, by Duflo's theorem [Du] $\overline{P} = \overline{P}(\alpha)$, which completes the proof of (ii).

For part (iii) we will show that the identity map $x \otimes y \mapsto x \otimes y$: $L(r) \otimes \overline{S} \longrightarrow L(r) \otimes \overline{S}^{\mathrm{Fr}}, x \in L(r), y \in \overline{S}$ is a U_ζ-isomorphism. This must be checked on the generators. The latter either belong to $\mathbf{u}_\zeta$ or to $\gamma(U)$. We'll do both cases.

Recall that U_ζ acts in $L(r) \otimes \overline{L}(\alpha)^{\mathrm{Fr}}$ via the pullback along $\rho = (I \otimes \mathrm{Fr})\Delta_\zeta$ with $U_\zeta \otimes U$ acting along the factors. Hence for an $a \in \mathbf{u}_\zeta$, in view of $\mathrm{Fr}(a) = \epsilon(a)$, we have $\rho(a)(x \otimes y) = ax \otimes y$ as needed. On the other hand, for a $\gamma(v), v \in U$ we have $\rho(\gamma(v)) = \gamma(v_1) \otimes v_2$ on account of γ being a U-comodule map. Further, $\gamma(v_1) \cdot x = \epsilon(\gamma(v_1))x$ for every $x \in L(r)$ because both $E^{(\ell)}$ and $F^{(\ell)}$ annihilate $L(r)$ for every restricted r (a rank 1 phenomenon). But then we see that $\rho(\gamma(v))(x \otimes y) = \epsilon(v_1)x \otimes v_2 y = x \otimes vy$ as needed.

We turn now to the last assertion in (iii). Fix two primitive vectors v^+ and w^+ in $L(r)$ and $\overline{L}(\alpha)$, respectively. By the preceding argument $v^+ \otimes w^+$ is a primitive vector in $L(r) \otimes \overline{L}(\alpha)^{\mathrm{Fr}}$. It also implies $K_\bullet(v^+ \otimes w^+) = Kv^+ \otimes w^+ = \zeta^r v^+ \otimes w^+$, where the "$\bullet$" denotes the action of U_ζ in $L(r) \otimes \overline{L}(\alpha)^{\mathrm{Fr}}$. As for the action of $\left[{K \atop \ell} \right]$, recall that $\left[{K \atop \ell} \right] = H - f$ in notation of Lemma 1. Hence $\rho(H) = [\rho(E^{(\ell)}, \rho(F^{(\ell)})] = 1 \otimes [e, f] = 1 \otimes h$, while $\rho(f) = f \otimes 1$. Thus $f_\bullet(v^+ \otimes w^+) = fv^+ \otimes w^+ = 0$, and therefore $H_\bullet(v^+ \otimes w^+) = v^+ \otimes hw^+ = \alpha v^+ \otimes w^+$. This shows that the weight of $v^+ \otimes w^+$ is $\lambda = (r, \alpha)$. It remains to note that by [Ja, 5.8.1] our module is irreducible. ∎

3. Decomposition of the U-module $\mathbf{u}_\zeta$

We need to review the structure of the PIMs (principal indecomposable modules) for $\mathbf{u}_\zeta$. For a start, we recall that by a theorem of Curtis–Lusztig [Lu 1, 2] (or directly) $\mathbf{u}_\zeta$ has ℓ-simple modules of the form $L(r)|_{\mathbf{u}_\zeta}$ with r ℓ-restricted. We let $P(r)$ denote the projective cover of $L(r)$ viewed as a $\mathbf{u}_\zeta$-module. We prefer to construct these modules by exploiting the comodule theory of the finitary dual $(U_\zeta)^0$ of U_ζ. We refer to Section 5 for a fuller discussion of this issue. Let $\mathbb{K}_\zeta[SL(2)]$ be the ζ-deformation of the coordinate algebra of $SL_2(\mathbb{K})$ ([CK 1]).

It turns out that $(U_\zeta)^0 \simeq \mathbb{K}_\zeta[SL(2)]$ (cf. §5). Green's theory [Gr] guarantees existence of the injective hull I_S for every right $\mathbb{K}_\zeta[SL(2)]$-comodule S. By [CK 1], or using the isomorphism just mentioned, we know that any simple comodule is isomorphic to $L(m)$ for some $m \in \mathbb{Z}^+$, treated as a right $\mathbb{K}_\zeta[SL(2)]$-comodule. Going in the opposite direction we can view $I(m){:}\ = I_{L(m)}$ as a left U_ζ-module. Restricting m to the interval $0 \leq r < \ell$ we arrive at the ℓ U_ζ-modules $I(r)$. Restricting $I(r)$ to $\mathbf{u}_\zeta$ we obtain the $P(r)$. This can be seen as follows. Thanks to [APW 2, 4.6; Li, 6.3] the restriction $I(r)|_{\mathbf{u}_\zeta}$ is the injective hull of $L(r)|_{\mathbf{u}_\zeta}$. As $\mathbf{u}_\zeta$ is Frobenius, $I(r)|_{\mathbf{u}_\zeta} = P(r)$. An alternate construction of PIMs can be found in [Su].

To describe the structure of the $P(r)$ we need to bring in a new family of U_ζ-modules. For an $m \in \mathbb{Z}^+$ we denote by $W(m)$ the U_ζ-module generated by a primitive vector v of weight m subject to conditions $F^{(j)}v = 0$ for $j > m$ and $F^{(i)}v \neq 0$ for all $i \leq m$. We refer to $W(m)$ as the mth **Weyl module** for U_ζ. The structure of $I(r)$ is as follows, [CK 1] or [Ch 2]. Let ρ be the reflection of r in $\ell - 1$, i.e., $\rho(r) = 2(\ell - 1) - r$. For every $r \neq \ell - 1$, $I(r)$ is characterized as the unique extension of $W(\rho(r))$ by $L(r)$, and $I(\ell - 1) = L(\ell - 1)$. Further $I(r), r \neq \ell - 1$, is a uniserial module with the factors $L(r), L(\rho(r)), L(r)$ in that order.

Put $r' = \ell - 2 - r$. Then $\rho(r) = r' + \ell$, and by the Steinberg–Lusztig theorem [Lu 1, 7.4] we have

$$L(\rho(r)) \simeq L(r') \otimes L(1)^{\mathrm{Fr}}.$$

Restricting to $\mathbf{u}_\zeta$ we get $L(\rho(r))|_{\mathbf{u}_\zeta} \simeq L(r') \oplus L(r')$. This yields in turn

$$P(r) \approx 2L(r) \oplus 2L(r'), \tag{3.1}$$

where $\approx$ signifies that two modules have the same composition factors. It follows readily that PIMs $P(r)$ and $P(s)$ are linked if and only if $s = r'$ or $s = r = \ell - 1$. Therefore, the block $\mathbf{u}_r$ of $\mathbf{u}_\zeta$ containing $P(r)$ has only one more PIM, namely, $P(r')$. Further, it is elementary that a PIM P associated with the absolutely simple module L has multiplicity $\dim L$ in its block. Thus $\mathbf{u}_r$ is the direct sum of $r + 1$ copies of $P(r)$ and $r' + 1$ copies of $P(r')$.

The next theorem requires an explicit decomposition of $\mathbf{u}_r$ into the direct sum of PIMs, and a special basis for $P(r)$.

We start by describing a special basis for $P(r)$. Let V be a nonsplit extension of $W(\rho(r))$ by $L(r)$. We recall that every finite-dimensional U_ζ-module is semisimple as a U_ζ^0-module [APW 1, §9].

Therefore $V|_{U^0_\zeta} \simeq L(r)|_{U^0_\zeta} \oplus W(\rho(r))|_{U^0_\zeta}$. It follows that V contains an element z_0 of weight r which is not in $W(\rho(r))$. The image $\overline{z}_0$ of z_0 in $L(r)$ is a primitive vector there, hence the set $\{F^{(i)}\overline{z}_0 | i = 0, \ldots, r\}$ forms a basis for $L(r)$. Note that $E\overline{z}_0 = F\overline{z}_r = 0$. Let's pull the $F^{(i)}\overline{z}_0$ into V by setting $z_i = F^{(i)}z_0$, $i = 0, \ldots, r$. Then Ez_0 and Fz_r lie in $W(\rho(r))$. Clearly their weights are $r+2$ and $-r-2$, respectively. Let v_0 be a primitive generator of $W(\rho(r))$, so that $\{v_i = F^{(i)}v_0 | i = 0, \ldots, \rho(r)\}$ is a basis of $W(\rho(r))$. Call such a basis **standard**. Then weight considerations lead to $Ez_0 = av_{\ell-r-2}$ and $Fz_r = bv_\ell$ for some $a, b \in \mathbb{K}$. These are "structure constants" of the extension. They depend on the choice of a standard basis for $W(\rho(r))$. We remark that neither a nor b is zero. For, the socle of $W(\rho(r))$ is the subspace spanned by $\{v_{r'+1}, \ldots, v_{\ell-1}\}$, and this is also the socle of V on account of V being nonsplit extension. Now, were $a = 0$, z_0 would be a primitive vector of V. Then z_0 would generate the submodule of V missing $\operatorname{soc} V$, a contradiction. Likewise, $b = 0$ implies the submodule $U_\zeta z_r$ misses $\operatorname{soc} V$. We mention in passing that in the standard basis generated by $E^{(r'+1)}z_0$ the structure constants equal $[\ell - r - 1]$ and $(-1)^r[\ell - r - 1]$, respectively. It follows that $I(r)$ is a unique nonsplit extension of $W(\rho(r))$ by $L(r)$.

The desired basis is this. Define $\rho(r)$ vectors w_k by the formulas

$$\begin{gathered} w_j = E^{(r'+1-j)}z_0, \quad j = 0, 1, \ldots, r', \quad w_{r'+1+i} = v_{r'+1+i}, \quad i = 0, 1, \ldots, r, \\ w_{\ell+i} = F^{(r+1+i)}z_0, \quad i = 0, 1, \ldots, r'. \end{gathered} \tag{3.2}$$

Thanks to the fact that $ab \neq 0$, the w_j are nonzero scalar multiples of v_j, hence they form a basis of $P(r)$

We proceed to the decomposition of $\mathbf{u}_r$. Let

$$B = \{0, 1, \ldots, (\ell-3)/2\} \quad \text{and} \quad \hat{B} = B \cup \{\ell - 1\}.$$

Recall the quantum Casimir element c_ζ. It can be checked that c_ζ acts on $L(r)$ by multiplication by the scalar

$$\lambda_r = \frac{\zeta^{r+1} + \zeta^{-(r+1)}}{(\zeta - \zeta^{-1})^2}.$$

One can see easily that $\lambda_r = \lambda_s$ iff $r + s = \ell - 2$. This limits the set of subscripts on λ_r to $\hat{B}$. Let

$$\Phi(x) = (x - \lambda_{\ell-1}) \prod_{r \in B} (x - \lambda_r)^2.$$

It turns out that $\Phi(x)$ is the minimal polynomial of c_ζ over $\mathbb{K}$. For, suppose that $f(x)$ is a polynomial annihilating c_ζ. Then from $f(c_\zeta)L(r) = 0$ it follows that

$f(\lambda_r) = 0$. This being the case for all $r \in \hat{B}$, $f(x)$ is divisible by $\prod_{r\in\hat{B}}(x-\lambda_r)$. In the same vein, $f(c_\zeta)P(r) = 0$ implies that $f(x)$ is divisible by $\Phi(x)$. On the other hand $\Phi(c_\zeta) = 0$, because $\Phi(c_\zeta)P(r) = 0$ for all $r \in \hat{B}$.

It follows immediately that the algebra $\mathbb{K}[c_\zeta]$ is isomorphic to the direct sum of algebras $\mathbb{K}[x]/((x-\lambda_r)^2), r \in B$ and $\mathbb{K}$. Let $\epsilon_r, r \in B, \epsilon_{\ell-1}$ be idempotents corresponding to summands isomorphic to $\mathbb{K}[x]/((x-\lambda_r)^2), r \in B$ and $\mathbb{K}$, respectively. The number of summands $\mathbf{u}_\zeta\epsilon_r$ of $\mathbf{u}_\zeta$ equals the number of blocks, so that each $\mathbf{u}_\zeta\epsilon_r$ is block algebra. A simple verification yields that ϵ_r annihilates every indecomposable $P(s)$ with $s \neq r, r'$. Thus $\mathbf{u}_\zeta\epsilon_r = \mathbf{u}_r$ for every $r \in \hat{B}$.

Recall the idempotents $e_j, j = 0, 1, \ldots, \ell-1$, defined in Remark 3. They give rise to the decomposition

$$\mathbf{u}_r = \bigoplus_{0\le j\le \ell-1} \mathbf{u}_r e_j.$$

In fact every $\mathbf{u}_r e_j$ is a PIM for $\mathbf{u}_r$, because, as we mentioned earlier, $\mathbf{u}_r$ is a direct sum of ℓ PIMs.

THEOREM 3: (1) *In the foregoing notation the $\mathbf{u}_\zeta\epsilon_r e_j$ are U-stable.*

(2) *Let $\mathbf{u}_r e_j$ be a summand of $\mathbf{u}_r$ isomorphic to $P(r)$. As a U-module $\mathbf{u}_r e_j$ is a direct sum of $2(r+1)$ copies of the trivial representation and $2(r'+1)$ copies of the defining representation of U, i.e.,*

$$\mathbf{u}_r e_j \simeq \bar{L}(0)^{2(r+1)} \oplus \bar{L}(1)^{2(r'+1)}.$$

Proof: (1) Since $E^{(\ell)}$ and $F^{(\ell)}$ commute with ϵ_r and e_j, this statement is clear.

(2) We start with a PIM P. Let N be the radical of $\mathbf{u}_\zeta$. Then NP is the maximal submodule W of P. Since W is not semisimple, N^2P is nonzero; hence it equals the socle of P.

For the remainder of the proof set $P = \mathbf{u}_r e_j$. By Lemma 3 in Section 2 N^2 is U-stable, therefore $N^2 e_j$ is U-stable. We conclude that $\operatorname{soc} P$ is U-stable. Next we note that, as $E^{(\ell)}$ and $F^{(\ell)}$ commute with K, for every U_ζ-module M, the K-weight subspace $M_\lambda, \lambda \in \mathbb{Z}_\ell$, is stable under U. Take $M = \operatorname{soc} P$. Then M_λ is 1-dimensional or zero. Therefore every vector of $\operatorname{soc} P$ is U-trivial, i.e., they are fixed points under the action of U. Moreover, the formula (3.1) shows that, for every weight λ, P_λ is 2-dimensional. Since the bottom composition factor has only one equal composition factor we see that the elements of P_λ, λ a weight of $\operatorname{soc} P$, are fixed by U. As $\operatorname{soc} P \simeq L(r)$ we get $r+1$ values of λ. This accounts for $2(r+1)$ copies of $\bar{L}(0)$ in $\mathbf{u}_r e_j$.

Suppose $\mathbf{u}_r e_j \simeq P(r)$ and recall the basis $w_j, j = 0, 1 \ldots, r'$ as in (3.2) above. It remains to show that none of the w_j are U-trivial. Let z_0 be a generator of weight r of $\mathbf{u}_r$. Then $w_j = E^{(p)}z_0$ for the p such that $j+p=r'+1$. Keeping in mind that z_0 is a fixed point we have

$$[F^{(\ell)}, w_j] = [F^{(\ell)}, E^{(p)}]z_0 = -\Big(\sum_{0\le i\le p} F^{(\ell-i)}\begin{bmatrix} K; 2i-\ell-p \\ i \end{bmatrix} E^{(p-i)}\Big) z_0.$$

We claim that all terms in the right-hand side of that formula with $i<p$ are zero.

For, suppose $p-i>0$. Up to a nonzero scalar $F^{(\ell-i)}E^{(p-i)}$ equals

$$F^{(\ell-p)}F^{(p-i)}E^{(p-i)}.$$

Further, by formula (1.1)

$$F^{(p-i)}E^{(p-i)}z_0 = g(c_\zeta, K)(c_\zeta - \lambda_r)z_0$$

for a polynomial $g(x,y)$. Also, it is immediate that $(c_\zeta-\lambda_r)z_0$ lies in $\operatorname{soc}\mathbf{u}_r e_j$. Now $F^{(\ell-p)}$ annihilates $\operatorname{soc}\mathbf{u}_r e_j$ because $\ell-p>r$ for every p.

For the $i=p$ term we have

$$\begin{aligned} -F^{(\ell-p)}\begin{bmatrix} K; p-\ell \\ p \end{bmatrix} z_0 = -\begin{bmatrix} r+p-\ell \\ p \end{bmatrix} F^{(r+1+j)}z_0 &= -\begin{bmatrix} -(j+1) \\ p \end{bmatrix} w_{\ell+j} \\ &= (-1)^{p+1}\begin{bmatrix} r'+1 \\ p \end{bmatrix} w_{\ell+j}. \end{aligned}$$

As $\left[\begin{smallmatrix} r'+1 \\ p \end{smallmatrix}\right] \neq 0$ we are done. ∎

As a by-product of the proof we note

COROLLARY: (1) *N^2 is generated by all $(c_\zeta-\lambda_r)\epsilon_r, r\in B$.*
(2) *N^2 consists of U-trivial elements.*

Proof: Both statements follow from the fact that N^2 is the sum of $\operatorname{soc} P$, over PIMs P, and the fact that $\operatorname{soc} P(r)$ is generated by $(c_\zeta-\lambda_r)z_0$, where z_0 is a weight r generator of $P(r)$. ∎

4. The center of U_ζ

We let $Z(A)$ denote the center of a $\mathbb{K}$-algebra A. We can use central idempotents ϵ_r to split U_ζ into the direct sum of subalgebras $\epsilon_r U_\zeta$. This leads to the splitting of $Z(U_\zeta)$ into the direct sum of $Z(\epsilon_r U_\zeta)$, which reduces the description of $Z(U_\zeta)$ to $Z(\epsilon_r U_\zeta)$.

The smash product representation of U_ζ implies that $\epsilon_r U_\zeta$ is isomorphic to $\mathbf{u}_r \# U$, where $\mathbf{u}_r = \epsilon_r \mathbf{u}_\zeta$ is a block of $\mathbf{u}_\zeta$. When $r = \ell - 1$, $\mathbf{u}_{\ell-1} = \operatorname{End} L(\ell-1) \simeq M_\ell(\mathbb{K})$. Let N be the radical of $\mathbf{u}_\zeta$ as before. Denote by $c = 4fe + (h+1)^2$ the usual Casimir element of U. Since $\mathbf{u}_{\ell-1} \cap N = 0$, we derive from Lemma 3 that $\epsilon_{\ell-1} U_\zeta \simeq M_\ell(\mathbb{K}) \otimes U \simeq M_\ell(U)$. It follows that $Z(\epsilon_{\ell-1} U_\zeta) = 1 \otimes Z(U) = 1 \otimes \mathbb{K}[c]$.

In what follows $r \neq \ell - 1$. We need to review a description of the center of $\mathbf{u}_r$. The result may be well-known, but we don't have a specific reference. The statement is that $Z(\mathbf{u}_r)$ is three-dimensional. We sketch a proof of it and also construct a basis of the center.

Let $\mathfrak{b}$ be the basic algebra of $\mathbf{u}_r$. As in ([Su]) $\mathfrak{b}$ is $\mathbb{K}$-algebra on a basis

$$\{e, c, a, b, e', c', a', b'\}$$

subject to relations

$$\begin{aligned} ec &= ce = c, \\ ab' &= ba' = c, \\ e'c' &= c'e' = c', \\ b'a &= a'b = c', \end{aligned}$$

e is a left unit of all non-primed generators and a right unit of all primed generators, e' is a left unit of all primed generators and a right unit of all non-primed generators, and all other products of basic elements are zero.

It is immediate from the definition of $\mathfrak{b}$ that $\{1_{\mathfrak{b}}, c, c'\}$ spans $Z(\mathfrak{b})$. On the other hand, it is straightforward to check that

$$\epsilon_r, \quad \epsilon_r(c_\zeta - \lambda_r) \quad \text{and} \quad \theta = \epsilon_r(c_\zeta - \lambda_r)\left(\sum_{j=0}^{r} e_{r-2j}\right) \tag{4.1}$$

lie in the center of $Z(\mathbf{u}_r)$ and are linearly independent. We conclude that the set (4.1) is a basis of $Z(\mathbf{u}_r)$. It follows readily that $M = \mathbb{K}(c_\zeta - \lambda_r)\epsilon_r \oplus \mathbb{K}\theta$ is the unique maximal ideal of $Z(\mathbf{u}_r)$

We proceed to the proof of

THEOREM 4: $Z(\epsilon_r U) = \mathbb{K}\epsilon_r \oplus (M \otimes \mathbb{K}[c])$.

Proof: In the easy direction we want to show that $(c_\zeta - \lambda_r)\epsilon_r \otimes \mathrm{k}[c]$ and $\theta \otimes \mathrm{k}[c]$ lie in $Z(\epsilon U)$. To this end we note that for every $x \in N^2, y \in U$, and $w \in \mathbf{u}_r$ we have $(xy)w = xwy + x[y,w] = xwy$, because by Lemma 3, $[y,w] \in N$ and $N^3 = 0$. Further, every $x \in Z(\mathbf{u}_r)$ commutes with every element of U, because it lies in $\mathbb{K}[c_\zeta, K]$. By Corollary of Theorem 3 the assertion follows.

We now turn to the harder part of the proof. Let $z = \sum x_i y_i, x_i \in \mathbf{u}_r, y_i \in U$ be a central element. We may assume that both sets $\{x_i\}$ and $\{y_i\}$ are linearly independent over $\mathbb{K}$. Then, as the y_i commute with K, $Kz = zK$ implies $Kx_i = x_i K$ for every i. But every element of $\mathbf{u}_\zeta$ that commutes with K lies in the subalgebra of $\mathbf{u}_\zeta$ generated by c_ζ and K. Further, this subalgebra centralizes U. It follows that the y_i lie in the center of U.

For the remainder of the proof we abbreviate $E^{(\ell)}$ to e, $F^{(\ell)}$ to f and we write h for H. Let $V = \mathbb{K}t \oplus \mathbb{K}v$ be a typical two-dimensional U-submodule occurring in decomposition of $\mathbf{u}_\zeta$ by Theorem 3(2). Then $V\#U$ is a $U-U$ subbimodule of U_ζ. Consequently, the left multiplication λ_u by a $u \in U$ induces a right U-linear map in $V\#U$. We denote by $M_u \in M_2(U)$ the matrix of λ_u relative to the basis $\{t, v\}$. We write M for M_c, hence M^n for M_{c^n}. We put

$$M^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

and denote by ω the standard automorphism of $\mathfrak{sl}_2$ defined by $\omega(e) = f, \omega(f) = e, \omega(h) = -h$. To proceed with the proof, we need

LEMMA 4: (1) *The elements of M^n are determined by a_n. We have $b_n = [f, a_n], c_n = \omega(b_n), d_n = \omega(a_n)$.*

(2) *a_n is a polynomial in c and h. Namely, $a_n = \sum_{i=0}^n c^i \phi_i(h)$ with $\phi_i(h)$ of degree ≤ 1. Further, $\phi_n(h) = 1$ and $\phi_{n-1}(h) = 2nh + n(2n+1)$.*

Proof: For calculations below one must keep in mind the action of e, f, h on t and v as well as the definition of c. The action is given by $[e,t] = 0, [f,t] = v, [h,t] = t$ and $[e,v] = t, [f,v] = 0, [h,v] = -v$. By definition we have $(*) c^n t = t a_n + v c_n$. We apply $\mathrm{ad}_L\, h$ and $\mathrm{ad}_L\, f$ in turn to both sides of $(*)$. For $\mathrm{ad}_L\, h$ we get $c^n t = t(a_n + [h, a_n]) + vg$ for some $g \in U$, which implies $[h, a_n] = 0$ on account of t, v being linearly independent over U. Thus a_n lies in the subalgebra of U generated by c and h, as claimed in the first part of (2). Further, $[f, c^n t] = c^n [f, t] = c^n v$. On the other hand, $[f, t a_n] + [f, v c_n] = t[f, a_n] + vg$ for a $g \in U$. It follows that $b_n = [f, a_n]$. As for the last claim in (1), it suffices to do the $n = 1$ case.

This can be computed directly. The result is

$$ct = t(c + 2h + 3) + 4ve \quad \text{and} \quad cv = 4tf + v(c - 2h + 3),$$

confirming the last part of (1).

The remaining assertions in (2) hold for $n = 1$ by the preceding two equations. The general case can be verified by induction on n. We leave the details to the reader. ∎

Conclusion of the proof of the Theorem: Let z be an element of $Z(\epsilon U)$. From the previous remarks we have

$$z = \sum_{i \leq n} x_i \cdot c^i, \quad x_i \in \mathbb{K}[c_\zeta, K].$$

The theorem's statement is equivalent to the claim that for every $i \geq 1$

($*$) x_i lies in the span of $(c_\zeta - \lambda_r)\epsilon$ and θ.

By the first part of the proof we know that if x_i satisfies ($*$), then $x_i c^i$ is in the center of U. Hence we may assume that the leading coefficient x_n of z doesn't satisfy ($*$). Next we compute $tz = zt$ in two ways. First, $tz = \sum tx_i c^i$. Second, by Lemma 4

$$zt = x_n tc^n + [x_n t(2nh + n(2n+1) + x_n v(4ne) + x_{n-1}t]c^{n-1} + \text{lower } c\text{-degree terms.}$$

Since the powers c^i are independent over $\mathbf{u}_r \otimes \mathbb{K}[h]$, we can equate the coefficients of c^n and c^{n-1} on both sides of the equation obtaining

(i) $$tx_n = x_n t,$$

(ii) $$tx_{n-1} = x_n t(2nh + (2n+1)n) + x_n v(4ne) + x_{n-1}t.$$

Now (i) implies that $x_n \in Z(\mathbf{u}_r)$. For, it shows that x_n commutes with t. Replacing t with v and repeating the argument above, we conclude that x_n commutes with v. On the other hand, if w is U-trivial, then $zw = wz$ clearly implies $x_n w = wx_n$. Since by Theorem 3, $\mathbf{u}_r$ has a basis whose elements fall into either of these two types, we get the assertion.

(ii) shows that $x_n t = x_n v = 0$, as we are working in the tensor product $\mathbf{u}_r \otimes U$. Also, by (i) we can write $x_n = a_0\epsilon + a_1(c_\zeta - \lambda_r)\epsilon + a_2\theta$. Taking into account $(c_\zeta - \lambda_r)t = \theta t = 0$, because both $(c_\zeta - \lambda_r)$ and θ lie in N^2 while $t, v \in N$ and $N^3 = 0$, we deduce $x_n t = a_0 \epsilon t = a_0 t = 0$. This forces $a_0 = 0$, a contradiction. ∎

We proceed to consider the block decomposition of U_ζ. Let ϵ be a primitive central idempotent of $\mathbf{u}_\zeta$. Then ϵU_ζ is a two-sided ideal of U_ζ. In fact, it is a block of U_ζ in the sense of the following

COROLLARY: *ϵU_ζ is an indecomposable algebra.*

Proof: By the theorem, the center of ϵU_ζ is a local algebra. ∎

5. Cofinite ideals of U

In this section we change our previous notation in that we are going to write U for U_ζ and H for $\mathbb{K}_\zeta[SL(2)]$. U^0 now stands for the finitary dual of U. The starting point of the section is existence of a Hopf pairing ([Ta 1]) between U and H. An equivalent formulation is existence of a Hopf map $\psi\colon H \to U^0$. The construction of ψ runs as follows (cf. [Ta 2], [deCL]). Let $L(1)$ be the 2-dimensional representation of U. It gives rise to the coordinate functions $c_{ij}, i, j = 1, 2$. Let A be the subalgebra of U^0 generated by the c_{ij}. Then A is a Hopf subalgebra of U^0 and the natural mapping $x_{ij} \to c_{ij}, i, j = 1, 2$, where x_{ij} are the standard generators of H [CK 1], extends to the Hopf algebra map $H \to A$. In fact, that map is an isomorphism [Ta 1], and moreover, according to [APW 1, Appendix] we have $A = U^0$. Below we will identify H with A.

Let π be the natural algebra homomorphism $U \to (U^0)^*$ [Sw, 6.0], $\pi(u)(\alpha) = \alpha(u), u \in U, \alpha \in U^0$. Note that the image of π is in $(U^0)^0$.

We compose π with $\psi^0\colon (U^0)^0 \to H^0$ to obtain $\phi = \psi^0\pi\colon U \to H^0$. Explicitly, $\phi(u)(x) = ((\psi^0\pi)(u))(x) = \pi(u)(\psi(x)) = (\psi(x))(u)$.

Using ψ we define a bilinear form

$$\langle , \rangle\colon U \otimes H \to \mathbb{K}, \quad \langle u, x\rangle = \psi(x)(u).$$

For a subspace L of U we set $L^\perp = \{x \in H | \langle L, x\rangle = 0\}$, and likewise for an $M \subset H$. We refer to the above as the annihilators of L and M, respectively. We recall that a subspace F of H^* is said to be dense in H^* if $F^\perp = 0$ in the natural pairing $H \times H^* \to \mathbb{K}$. A subspace L of U (M of H) is said to be closed if $L^{\perp\perp} = L (M^{\perp\perp} = M)$. A pairing $U \times H \to \mathbb{K}$ is nondegenerate on the left (right) if $H^\perp = 0$ ($U^\perp = 0$). A pairing is nondegenerate if it is both left and right nondegenerate.

LEMMA 5: *Every cofinite ideal of U and every finite-dimensional subspace of H is closed.*

Proof: Let $I^{\perp *}$ stand for the annihilator of I in U^*. In view of $H = U^0$ we have $I^{\perp *} = I^\perp$. But then $I^{\perp\perp} = I^{\perp * \perp}$, which is I by [Sw, A.1].

The second statement is well-known for a nondegenerate pairing [Ab, 2.2]. The same proof works for a right nondegenerate pairing. It remains to note that our pairing is right nondegenerate by the equality $H = A$. ∎

We let $\Lambda_{\mathrm{cof}}(U)$ and $\Lambda^{\mathrm{fin}}(H)$ denote the lattices of cofinite ideals of U and finite-dimensional subcoalgebras of H, respectively.

LEMMA 6: (1) *For every ideal I of U and every subcoalgebra C of H, $I^{\perp}$ is a subcoalgebra of H and $C^{\perp}$ is an ideal of U*

(2) *$\Lambda_{\mathrm{cof}}(U)$ is antiisomorphic with $\Lambda^{\mathrm{fin}}(H)$ under $I \mapsto I^{\perp}$.*

Proof: (1) The usual proof ([Sw, 1.4.3]) of the assertions works here thanks to the property $\phi(U)$ dense in H^*, which is equivalent to $U^{\perp} = 0$

(2) It is straightforward to see that the mappings $I \mapsto I^{\perp}$ and $C \mapsto C^{\perp}$ set up two inclusion reversing correspondences between $\Lambda_{\mathrm{cof}}(U)$ and $\Lambda^{\mathrm{fin}}(H)$. Lemma 5 makes it clear that those maps are mutual inverses of each other. By general principles they send unions to joins and conversely. ∎

For a coalgebra C we let C^C denote the right regular C-comodule. We put $E = \mathrm{End}^C(C^C)$ for the algebra of all C-endomorphisms of C^C. We recall that C has the natural structure of the right C^*-module via $c \leftharpoonup c^* = \sum \langle c_1, c^* \rangle c_2$.

The next very useful lemma is a variation on a well-known statement in the theory of coalgebras. A weaker form can be found in [Ho, p. 18].

LEMMA 7: *E is antiisomorphic with the algebra C^* acting on C by the right 'hits'.*

Proof: In one direction, given $f \in E$ we associate with it $\phi = \epsilon \circ f \in C^*$. In the opposite direction, we send $\phi \in C^*$ to f_ϕ: $x \mapsto x \leftharpoonup \phi$. One can check that $f_\phi \in E$ and the above-defined maps are mutually inverse antiisomorphisms between E and the image of C^* in E. It remains to note that by [Sw, 9.1.2] $\phi \mapsto f_\phi$ is an injection. ∎

A description of the lattice $\Lambda^{\mathrm{fin}}(H)$ begins with the representation of H as a direct sum of injective indecomposables. We view H as a coalgebra only. We know that every simple right H-comodule is $L(r)$ for some r by [CK 1] (or using the fact that $H = U^0$ and the classification of the simple finite-dimensional U_ζ-modules). Let $I(r)$ be the injective hull of $L(r)$, and put $m(r) = \dim L(r)$. According to J. Green [Gr, 1.5] we have

$$H = \bigoplus I(r)^{m(r)}, \tag{5.1}$$

where $m(r)$ is the multiplicity of $I(r)$ in H, because the $L(r)$ are absolutely simple. Decomposition (5.1) is unique up to isomorphism; see [Gr, 1.5g].

We refer to the representation (5.1) for a coalgebra C as **the Green decomposition** for C.

We denote by $I(r)^{(j)}, j = 1, \ldots, m(r)$, the j-th copy of $I(r)$ in H. Let C be a subcoalgebra of H. We want to connect the structure of a subcoalgebra to that of injective comodules. The first step is

PROPOSITION 2: (i) $C = \bigoplus_r (\bigoplus_j C \cap I(r)^{(j)})$.
(ii) *The set of subcomodules* $C \cap I(r)^{(1)}$ *determines* C.

Proof: (i) holds iff every component of an $x \in C$ lies in C. Hence assume $x = \sum x_{r,j}$ and let $\pi_{r,j}$ be the projection of H on $I(r)^{(j)}$. By Lemma 7 we can regard $\pi_{r,j}$ as an element of H^*. As $C \leftharpoonup \pi_{r,j} \subset C$, the proof is complete.

(ii) It suffices to show that for every isomorphism ϕ: $I(r)^{(1)} \simeq I(r)^{(j)}$, $(C \cap I(r)^{(1)})\phi = C \cap I(r)^{(j)}$. Since ϕ can be lifted to an H-endomorphism of H, we can view ϕ as an element of H^*. But then $(C \cap I(r)^{(1)})\phi \subset C \cap I(r)^{(j)}$ for the same reason as above. Since ϕ has an inverse $I(r)^{(j)} \to I(r)^{(1)}$, the equality follows. ∎

We can improve significantly on the above proposition. We need to introduce some notation and terminology. Given a right H-comodule M we say that a subcomodule N of M is **fully invariant** if it is stable under the action of $\mathrm{End}^H(M)$. In the case $M = H$ denote by ${}^HN$ and $C(N)$ the left H-comodule and the subcoalgebra generated by N, respectively. Of course, $C(N) = {}^HN$. We let Λ_r denote the sublattice of the fully invariant subcomodules of $I(r)$. Let $\prod \Lambda_r$ be the Cartesian product of the Λ_r and put φ_r for the natural projection $\prod \Lambda_r \to \Lambda_r$. Set $E_{r,s} = \mathrm{Hom}^H(I(r), I(s))$ and call an element $(X_r)_{r \in \mathbb{Z}^+} \in \prod \Lambda_r$ **balanced** if

$$X_r E_{r,s} \subset X_s \quad \text{for all } r, s.$$

Let $\Lambda^{\mathrm{coalg}}(H)$ be the lattice of the subcoalgebras of H. Also, in what follows we drop the superscript and write $I(r)$ for $I(r)^{(1)}$.

PROPOSITION 3: *The mapping* θ: $C \mapsto (C \cap I(r))_{r \in \mathbb{Z}^+}$ *is a lattice embedding* $\Lambda^{\mathrm{coalg}}(H) \to \prod \Lambda_r$. *The image of* θ *is the set of all balanced elements of* $\prod \Lambda_r$ *and* $\varphi_r \theta$ *is a surjection for all* r.

Proof: As in the proof of Proposition 2, for every $\phi \in E_{r,s}$ we have $(C \cap I(r))\phi \subset C \cap I(s)$. Thus $C \cap I(r) \in \Lambda_r$ for every r and also the sequence $\theta(C)$ is balanced.

The rest of the proposition is based on several observations. Let π_r be the projection of H on $I(r)$. Then the following holds: (i) $C \cap I(r) = C\pi_r$, (ii) for every $X \in \Lambda_r, C(X)\pi_r = X$, and (iii) $C(X)\pi_s = XE_{r,s}$.

(i) is obvious. As for (ii), it suffices to show that $C(X)\pi_r \subset X$. We have $C(X)\pi_r = ({}^H X)\pi_r = XH^*\pi_r$. Pick a $\phi \in H^*$. Then $\phi\pi_r$ maps X into $I(r)$. By injectivity of $I(r)$, $\phi\pi_r$ is the restriction of a $\psi \in \mathrm{End}(I(r))$. As X is fully invariant, (ii) follows.

For (iii) we have $C(X)\pi_s = XH^*\pi_s$. As $H^*\pi_s|_{I(r)} = E_{r,s}$, we are done.

Now, pick two subcoalgebras C and D. For every $r \in \mathbb{Z}^+$ we have

$$(C + D) \cap I(r) = (C + D)\pi_r = C\pi_r + D\pi_r = C \cap I(r) + D \cap I(r)$$

which proves that θ is a lattice map.

Further, suppose $(X_r)_{r\in\mathbb{Z}^+}$ is a balanced sequence. Put $C = \sum C(X_r)$. Then $C\pi_r = \sum C(X_s)\pi_r = X$ on account of (ii) and (iii). Also, (ii) shows that $\varphi_r\theta$ is a surjection. ∎

The last proposition points to the importance of knowing fully invariant subcomodules of $I(r)$. It will turn out that these coincide with the subcomodules of $I(r)$. Thus we proceed to a description of the subcomodule lattice of $I(r)$. In this we follow [CK 1, Thm. 5.2]; also see [Ch 2].

There is a symmetry at work here expressed, in general, by the ℓ-affine Weyl group (cf. [Th]). In the case at hand we will use instead a bijection $\rho: \mathbb{Z} \to \mathbb{Z}$, called an ℓ-reflection. Write $m = m_0 + m_1\ell$, where $0 \le m_0 \le \ell - 1$. Define $\rho(m) = m$, if $m_0 = \ell - 1$ (i.e., m is a Steinberg weight). If $m_0 \ne \ell - 1$, then set $\rho(m)$ equal to the reflection of m in the nearest Steinberg weight to the left. All in all we have

$$\rho(m) = \ell - 2 - m_0 + (m_1 - 1)\ell \quad \text{if } m_0 \ne \ell - 1, \quad \text{and } \rho(m) = m \text{ otherwise.}$$

Next, we need to bring in a new family of H-comodules $\{M(r) | r \in \mathbb{Z}^+\}$. By definition $M(r)$ is the (contragredient) dual of $W(r)$, i.e., $M(r) = W(r)^*$. We note an independent construction of this family. Let $A = \mathbb{K}_\zeta[e_1, e_2]$ be the ζ-plane, that is the algebra generated by e_1, e_2 subject to one relation $e_2e_1 = \zeta e_1e_2$. A is a $\mathbb{Z}^+$-graded algebra with the n-th component A_n consisting of all homogeneous polynomials of degree n; A is also a natural right H-comodule algebra via

$$\omega(e_j) = \sum e_k \otimes x_{kj}.$$

Each A_n is a subcomodule and the family $\{A_r\}$ coincides with $\{M(r)\}$.

We now state the main facts about $I(r)$ from [CK 1] (see also [Ch 2]).

PROPOSITION 4: (1) *$I(r)$ is a local, self-dual comodule.*

(2) *If $r = \ell - 1 + r_1\ell$ is a Steinberg weight, then $I(r) = L(r)$.*

(3) *If $r = r_0 + r_1\ell$ with $r_0 \neq \ell - 1$ and $r_1 > 0$, then nonzero proper subcomodules of $I(r)$ are precisely $M(r)$, $W(\rho^{-1}(r))$, $L(r)$.*

(4) *If $r = r_0$, the nonzero proper subcomodules of $I(r)$ are precisely $W(\rho^{-1}(r))$ or $L(r)$.*

We proceed to a description of "links" between the $I(r)$.

LEMMA 8: $\dim \mathrm{Hom}^H(I(r), I(s)) \leq 1$ *for all r, s. Equality holds iff $r = \rho(s)$ or $s = \rho(r)$.*

Proof: It is well-known [Lu 1] that $W(m)$ has two composition factors $L(m)$ and $L(\rho(m))$. From Proposition 4 we see that $I(r)$ and $I(s)$ have a common factor iff r and s are as stated in the lemma.

Suppose $r = \rho(s)$ for definiteness. The preceding proposition makes it clear that the multiplicity of $L(s)$ in $I(r)$ is 1. Hence the dimension in question is at most 1. Now dualize $0 \to W(s) \to I(r)$ to get $I(r)^* \to W(s)^* \to 0$. As $I(r)$ is self-dual, we are done. ∎

LEMMA 9: *Let $r = \rho(s)$ and f_r, g_s be non-zero elements of $\mathrm{Hom}^H(I(r), I(s))$ and $\mathrm{Hom}^H(I(s), I(r))$, respectively. Then $\mathrm{Ker}\, f_r = M(r)$ and $\mathrm{Ker}\, g_s = W(\rho^{-1}(s))$.*

Proof: We only consider f_r. From the previous lemma, the image of f_r is $M(s)$. Since $\mathrm{soc}\, M(s) = L(s)$, we have $f_r(M(r)) = 0$. As the codimension of $M(r)$ equals $\dim M(s)$, the assertion follows. ∎

We combine Propositions 3 and 4 with the last lemma to derive the structure of an arbitrary finite-dimensional subcoalgebra of H.

We say that a coalgebra is local if it has a unique maximal subcoalgebra. Similarly, a comodule is local if it has a unique maximal subcomodule.

THEOREM 5: *The following properties hold for $\Lambda^{\mathrm{fin}}(H)$*

1. *The lattice $\Lambda^{\mathrm{fin}}(H)$ is distributive.*
2. *Every subcoalgebra is a unique sum of local subcoalgebras.*
3. *Every local subcoalgebra has the form $C(X)$, where X is a local subcomodule of an $I(r)$.*
4. *For a coalgebra $C = C(X)$, where X is a subcomodule of $I(r)$, the Green decomposition is*

$$C = X_{\rho(r)}^{m_{\rho(r)}} \oplus X_r^{m_r} \oplus X_{\rho^{-1}(r)}^{m_{\rho^{-1}(r)}}$$

with $X_{\rho^{\pm 1}(r)} = X_r \operatorname{Hom}^H(I(r), I(\rho^{\pm 1}(r))$

Proof: (1) Proposition 4 makes it evident that the lattice of subcomodules of every $I(r)$ is distributive. Then, by a result in [Ste, Cor. 1 of Thm. 4.1], we conclude that every subcomodule is fully invariant. As the class of distributive lattices is closed under taking of Cartesian products and sublattices, $\Lambda^{\text{fin}}(H)$ is distributive.

(2) This is a standard fact in the theory of distributive lattices [DP].

(3) In one direction, assume $C = C(X)$ for a local $X \subset I(r)$. Were $C = D+E$ for some proper subcoalgebras D and E, we would have by Proposition 3

$$X = C \cap I(r) = (D+E) \cap I(r) = D \cap I(r) + E \cap I(r),$$

a contradiction.

Conversely, pick a local C. Let $X_j = C \cap I(j)$. Then C and $\sum C(X_j)$ have equal intersection with every $I(r)$ by Proposition 3, hence they are equal by Proposition 2. Thus $C = C(X_j)$ for some X_j. This forces X_j to be local, for were $X_j = X_j' + X_j''$ with proper X_j', X_j'', so would C be a sum of proper subcoalgebras.

(4) As $L(r)$ is the socle of $I(r)^{(j)}$, $C \cap I(r)^{(j)}$ is an indecomposable subcomodule of C. Proposition 2(i) gives C-injectivity of these and the Green decomposition of C. Further, properties (i)–(iii) in the proof of Proposition 3 say that $C(X) = \bigoplus_s (XE_{r,s})^{m_s}$. But by Lemma 8, $E_{r,s} \neq 0$ iff $s = \rho(r)$ or $s = \rho^{-1}(r)$. ∎

We pass on to a classification of the cofinite ideals. A few preliminaries are in order. Given an H-comodule M, $\operatorname{ann}_U M$ is the annihilator of M in U, where M is regarded as a U-module via the duality $\langle , \rangle$. For an ideal N of U and a U-module I, $\operatorname{ann}_I N$ denotes the submodule $\{x \in I | Nx = 0\}$.

For an H-comodule M, $\operatorname{cf}(M)$ denotes the **coefficient space** of M defined in [Gr].

LEMMA 10: *If M is a subcomodule of H, then* $\operatorname{cf}(M) = C(M)$, *and* $C(M)^{\perp} = \operatorname{ann}_U M$

Proof: By the definition of $\operatorname{cf}(M)$ one has $\operatorname{cf}(M)^{\perp} = \operatorname{ann}_U M$.

By ([Gr, 1.2f]), $\operatorname{cf}(M)$ is a subcoalgebra containing M, whence $C(M) \subset \operatorname{cf}(M)$. Conversely, pick $m \in M$ and let $\Delta(m) = \sum m_i' \otimes m_i''$. Then $m \leftharpoonup m_i^* = m_i'' \in C(M)$ for every i. But, by definition, $\operatorname{cf}(M)$ is spanned by the various m_i'' as m runs over a basis for M. ∎

We exploit the duality between H and U to obtain

THEOREM 6: *Let X run over the set of injective indecomposable H-comodules and their local subcomodules.*

Any cofinite ideal of U is a unique intersection of the ideals $\operatorname{ann}_U X$. ∎

Our next goal is to give the structure of the ideals $\operatorname{ann}_U X$ of the previous theorem.

Two weights r and s are called **linked** if $r = \rho(s)$ or $r = \rho^{-1}(s)$. The linkage generates an equivalence relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$. The class of r, $\operatorname{bl}(r)$, is called the block of r. If r is not a Steinberg weight, i.e., $r_0 \neq \ell - 1$, then $\operatorname{bl}(r) = \operatorname{bl}(m)$, where m is the smallest weight in $\operatorname{bl}(r)$. Such an m is necessarily restricted, that is, $0 \leq m \leq \ell - 1$. The set $\operatorname{bl}(r)$ equals $\{\rho^{-i}(r_0) | i \in \mathbb{Z}^+\}$. We remark that $\bigoplus_{s \in \operatorname{bl}(r)} I(s)^{d_s}$ is exactly the block subcoalgebra summand of H containing $I(r)$, whence our notation. We may ignore all injectives not in the block of $I(r)$. For those in the block we write $r_j = \rho^{-j}(r_0)$ and $I(j) = I(r_j)$.

LEMMA 11: *Let N be a cofinite ideal of U, $C = N^\perp$ and I one of the $I(j)$. Put $X = C \cap I$. Then $X = \operatorname{ann}_I N$.*

Proof: For a subcomodule $Y \subset I$, $NY = 0$ iff $N \subset \operatorname{cf}(Y)^\perp$. Therefore $C = N^\perp \supset \operatorname{cf}(Y)^{\perp\perp} = \operatorname{cf}(Y)$. However, $\operatorname{cf}(Y) = C(Y)$ and therefore $C \supset C(Y)$, which implies $X \supset Y$ by Proposition 3. Thus X is the largest subcomodule of I annihilated by N. ∎

Let X be as in Theorem 6 and let

$$X = X_1 \supset X_2 \supset \cdots \supset X_t \supset 0 \quad (t \leq 4) \tag{5.2}$$

be a composition series for X. Let $\{L_i\}$ be the set of composition factors of (5.2) in order from top to bottom. Put $C_i = \operatorname{cf}(L_i)$ and $\mathfrak{m}_i = \operatorname{ann}_U L_i$.

Recall that $A \wedge B$ denotes the wedge [Sw, 9.0] of two subspaces A and B.

THEOREM 7: *In the foregoing notation*

$$C(X) = C_t \wedge \cdots \wedge C_1 \quad \textit{or equivalently} \quad \operatorname{ann}_U X = \mathfrak{m}_t \cdots \mathfrak{m}_1.$$

Proof: Recall that $\mathfrak{m}_i = C_i^\perp$ by Lemma 10, while from [Sw, 9.0.0(b)], which holds for our pairing as well, we derive $(C_t \wedge \cdots \wedge C_1)^\perp = \mathfrak{m}_t \cdots \mathfrak{m}_1$.

Let $N = \mathfrak{m}_t \cdots \mathfrak{m}_1$ and put $D = N^\perp$. We will prove the equality $D = C(X)$ by showing that $D \cap I(k) = C(X) \cap I(k)$ for all k (see Proposition 2). The original Hopf pairing $U \times H \to \mathbb{K}$ induces the pairing $U/N \times D \to \mathbb{K}$, which is clearly nondegenerate. It follows that the algebras D^* and U/N are isomorphic. As $\cap \mathfrak{m}_i$ is the radical of U/N, we conclude that

$$\operatorname{corad}(D) = (\operatorname{rad} D^*)^\perp = \sum \mathfrak{m}_i^\perp = \sum C_i.$$

Suppose $X \subseteq I(j)$. Apart from the trivial case $X = L(j)$, X can be equal to $M(j), W(j+1)$ or $I(j)$. The first two cases are similar. Thus we consider two cases.

(1) Let $X = M(j), j > 0$. By Theorem 5(4) and Lemmas 8 and 9,

$$C(X) \cap I(j-1) = L(j-1) \quad \text{and} \quad C(X) \cap I(k) = 0 \quad \text{for } k \neq j-1, j.$$

On the other hand, whenever $D \cap I(k) \neq 0$, $D \supset \operatorname{soc} I(k) = L(k)$. But $L(k)$ is in the coradical of D iff $k = j-1, j$. Thus we reduce to two cases.

(i) Suppose $D \cap I(j) \supsetneqq M(j)$; then $D \supseteqq W(j+1)$. Since $L(j+1)$ is a composition factor of $W(j+1)$, but doesn't lie in the corad(D), we arrive at a contradiction.

(ii) Assume $D \cap I(j-1) \supsetneqq L(j-1)$. Then by the same argument as in (i), $D \cap I(j-1) \subseteqq W(j)$. Suppose we have equality there. Using Lemma 11 we would have $NW(j) = 0$. Now $N = \mathfrak{m}_j \mathfrak{m}_{j-1}$ and $\mathfrak{m}_{j-1} L(j) = L(j)$ on account of $\mathfrak{m}_{j-1} + \mathfrak{m}_j = U$. But $L(j)$ is the top composition factor of $W(j)$, hence $NW(j) = \mathfrak{m}_j W(j) = 0$, which is impossible, since $W(j)$ is not semisimple. This completes case (1).

(2) Let $X = I(j)$ for a $j \neq 1$. Combining Theorem 5 and Lemmas 8 and 9 we get $C(X) \cap I(j-1) = W(j)$ (for $j > 0$) and $C(X) \cap I(j+1) = M(j+1)$.

On the other hand, $L(k)$ is in the socle of D iff $k \in \{j-1, j, j+1\}$. Therefore $D \cap I(k) \neq 0$ for those k only. Were $D \cap I(k) \supsetneqq C(X) \cap I(k)$ then D would have a simple subcomodule $L(s)$ with $s \notin \{j-1, j, j+1\}$, a contradiction.

The last case to consider is when $X = I(1)$. Here we must dispose of the possibility $D \cap I(0) = I(0)$. If this holds, then $NI(0) = 0$ follows. Now by definition $N = \mathfrak{m}_1 \mathfrak{m}_0 \mathfrak{m}_2 \mathfrak{m}_1$; hence $NI(0) = (\mathfrak{m}_1 \mathfrak{m}_0) I(0)$ on account of $\mathfrak{m}_i L(0) = L(0)$ for $i = 1, 2$. Further, $\mathfrak{m}_0 I(0) = W(1)$, for otherwise $I(0)/L(0)$ would be a semisimple U-module, a contradiction. It follows that $\mathfrak{m}_1 W(1) = 0$, which is again impossible, since $W(1)$ is not semisimple. We conclude that $D \cap I(0) = W(1)$, and the proof is complete. ∎

Remark 4: The annihilator in case (1) in the above proof is a particular instance of the annihilator of an R-module E representing a non-zero element of $\operatorname{Ext}^1_R(Y, X)$ where X and Y are simple finite-dimensional R-modules. Denote by $\mathfrak{m}_X$ and $\mathfrak{m}_Y$ the annihilators in R of those simple modules. The following holds: $\operatorname{ann}_R E = \mathfrak{m}_X \mathfrak{m}_Y$ if and only if $\dim \operatorname{Ext}^1_R(Y, X) \leq 1$. This is the case, of course, for every two U-simples.

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